DECOMPOSITION OF POLYTOPES USING INNER PARALLEL BODIES

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Abstract. Let P be a polytope and P_λ, $\lambda \leq 0$ an inner parallel body of P, i.e., the polytope constructed by moving the facets of P inwards at distance $|\lambda|$. We study when P_λ is a summand of P. We characterize the polytopes P satisfying that P_λ is a summand of P_μ with $\lambda < \mu \leq 0$. Besides, we provide an explicit decomposition of such polytopes using the so called form bodies of their inner parallel bodies.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets in the Euclidean space $\mathbb{R}^n$. Let B_n be the n-dimensional unit ball and S^{n-1} the corresponding unit sphere. The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}_n(M)$ and its closure by $\mathrm{cl}(M)$. For $K \in \mathcal{K}^n$ and $u \in \mathrm{S}^{n-1}$, $h(K,u) = \sup\big\{\langle x, u\rangle : x \in K\big\}$, denotes the *support function* of the set $K \in \mathcal{K}^n$ (see e.g. [11, s. 1.7]).
Let $K, L \in \mathcal{K}^n$. L is called a summand of K if there exists $M \in \mathcal{K}^n$, such that $K = L + M$, where $+$ refers to the usual Minkowski or vectorial addition of sets in $\mathbb{R}^n$.
Since we mainly consider polytopes, we now introduce the necessary notions only for polytopes in order to state our main result. All these notions exist for arbitrary convex bodies and are introduced in Section 2. Let $u_i \in \mathrm{S}^{n-1}$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$ and $P \in \mathcal{K}^n$ be the polytope

$$P = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leq b_i, 1 \leq i \leq m\}$$

with outer normal vectors u_i, $1 \leq i \leq m$ to the facets, i.e., $n-1$-faces of P. Thus none of the inequalities $\langle x, u_i\rangle \leq b_i$, $1 \leq i \leq m$ is redundant. Then, for $-\mathrm{r}(P) \leq \lambda \leq 0$, the inner parallel body of P at distance $|\lambda|$ is defined as

$$P_\lambda = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leq b_i - |\lambda| \ , \ 1 \leq i \leq m\};$$

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i.e., P_λ is the polytope which arises *by moving* inwards the facets of P all at distance $|\lambda|$. Here $\mathrm{r}(P)$ denotes the inradius of P (see Section 2 for the precise definition). It is natural to ask whether P can be retrieved through the Minkowski sum of P_λ with another convex body, which obviously, should be also a polytope. In this paper we study when the inner parallel bodies P_λ, for $-\mathrm{r}(P) \leq \lambda \leq 0$, of a given polytope P are summands of P. An important role in some partial answers plays the so-called *form body*, that in the case of the polytope P, is

$$P^{\mathbb{1}} = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 1, 1 \leq i \leq m\}.$$

By $\mathcal{U}(P)$ we denote the set of vectors $u \in \mathrm{S}^{n-1}$ that are normal to facets. Next we deal with our main results.

Theorem 1.1. *Let P be a polytope with inradius $\mathrm{r}(P)$. P_τ is a summand of P_μ for all $\tau \leq \mu \leq 0$ if and only if*

$$h(P, u) = h(P_\tau, u) + \int_\tau^0 h((P_\mu)^{\mathbb{1}}, u) d\mu,$$

for all $u \in \mathrm{S}^{n-1}$.

Moreover, unlike what happens in dimension 2 (see Section 4), there exist polytopes P all whose inner parallel bodies P_λ, $-\mathrm{r}(P) \leq \lambda \leq 0$, are summands of them, but P_λ is not a summand of P_μ for $\lambda < \mu < 0$.
Furthermore, we provide sufficient and necessary conditions, relying only on the facets of $P_\tau + P_\tau^{\mathbb{1}}$, in order P_τ to be a summand of all P_μ for $\tau \leq \mu \leq 0$.

Theorem 1.2. *Let P be a polytope and $-\mathrm{r}(P) \leq \tau \leq 0$. P_τ is a summand of P_μ for all $\tau \leq \mu \leq 0$ if and only if $\mathcal{U}(P_\mu + P_\mu^{\mathbb{1}}) = \mathcal{U}(P_\mu)$ for $\tau \leq \mu \leq 0$.*

A fundamental tool to address this problem is a criterion of Shephard (see [12]) that characterizes the polytopes which can be summands of a given one. This result has been generalized and proven to be equivalent to conditions having very different flavor, as intersections of translates or monotonicity of mixed volumes. There is a vast amount of literature dealing with decomposition of convex bodies. For a complete description of the situation we refer to Schneider [11, Section 3.2, Notes to Section 3.2] and the references therein.
The problem of studying when inner parallel bodies of a polytope are summands of it makes also sense for an arbitrary convex body K. However, the tools we have used in this work for polytopes seem not to work for general convex bodies, except for dimension 2, as we shall see in Section 4.
The paper is organized as follows. In Section 2 we introduce the notions and results which are needed. In Section 3 we collect general results on inner parallel bodies and summands of polytopes. We devote Section 4 to dimension 2, where the results are not restricted to polytopes. Finally in Section 5 we prove Theorems 1.1 and 1.2 as a consequence of a more general result, namely, Theorem 5.2, and give examples for how the situation differs from dimension 2.

2. Background

In this section we introduce all further necessary notions. We provide also for arbitrary convex bodies the corresponding notions already introduced for polytopes. The Minkowski difference of two sets $K, L \subset \mathbb{R}^n$ is defined as follows:

$$K \sim L = \{x \in \mathbb{R}^n \,:\, x + L \subseteq K\}.$$

We notice that $(K \sim L) + L \subset K$ and inequality may be strict.

Let $K \in \mathcal{K}^n$. The *inradius* $\mathrm{r}(K)$ of K is the radius of one of the largest balls which fit inside K, i.e.,

$$\mathrm{r}(K) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\, \mathrm{B}_n \subset K\}.$$

For $-\mathrm{r}(K) \leq \lambda \leq 0$ the *inner parallel body of K at distance* $|\lambda|$ is the Minkowski difference of K and $|\lambda|\, \mathrm{B}_n$, i.e.,

$$K_\lambda := K \sim |\lambda|\, \mathrm{B}_n = \{x \in \mathbb{R}^n : |\lambda|\, B_n + x \subset K\} \in \mathcal{K}^n\,.$$

Notice that $K_{-\mathrm{r}(K)}$ is the set of incenters of K, usually called *kernel* of K. The dimension of $K_{-\mathrm{r}(K)}$ is strictly less than n (see [2, p. 59]). The inner parallel body of K can be defined, equivalently (see [11, Section 3.1]), as

$$K_\lambda = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K,u) - |\lambda|\,, u \in \mathrm{S}^{n-1}\}.$$

For $K \in \mathcal{K}^n$, $u \in \mathrm{S}^{n-1}$ and $H(K,u) = \{x \in \mathbb{R}^n : \langle x, u\rangle = h(K,u)\}$ the supporting hyperplane to K with outer normal vector u, we denote by $F(K,u) = K \cap H(K,u)$ the corresponding face of K cut off by $H(K,u)$. A vector $u \in \mathrm{S}^{n-1}$ is a 0-extreme normal vector of K, if it cannot be written as linear combination of two linearly independent normal vectors at the same boundary point of K. Thus, u is called 0-extreme, if it is the unique normal vector to $F(K,u)$. Hence, as for polytopes, the set of 0-extreme vectors if K is denoted by $\mathcal{U}(K)$. Minkowski's Theorem ([11, Corollary 1.4.5]) yields that

$$K = \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle \leq h(K,u), u \in \mathcal{U}(K)\},$$

and thus the inner parallel body in fact is

$$K_\lambda = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K,u) - |\lambda|\,, u \in \mathcal{U}(K)\}.$$

We also point out that the latter definition of inner parallel body is consistent with the notion of inner parallel body of a polytope provided in the introduction.

The *form body* of a convex body $K \in \mathcal{K}^n$, denoted by $K^{\mathbb{1}}$, is defined as (see e.g. [3])

$$K^{\mathbb{1}} = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 1, u \in \mathcal{U}(K)\}.$$

In the following we collect some properties of inner parallel bodies, form bodies and extreme vectors which will be needed later on. They can be found in [9, Lemmas 2.4, 2.6, 4.4 and 4.5].

Lemma 2.1. *Let $K, L \in \mathcal{K}^n$ and $-\mathrm{r}(K) < \lambda \leq 0$. The following facts hold:*

(i) $\mathcal{U}(K_\lambda) \subseteq \mathcal{U}(K)$

(ii) $\mathcal{U}(K \sim L) \subseteq \mathcal{U}(K)$

(iii) $\mathrm{cl}(\mathcal{U}(K)) = \mathcal{U}(K^{\mathbb{1}})$

(iv) $\mathcal{U}(K) \cup \mathcal{U}(L) \subseteq \mathcal{U}(K+L)$

(v) $h(K_\lambda, u) = h(K,u) - |\lambda|\, h(\mathrm{B}_n, u) = h(K,u) + \lambda$, *for* $u \in \mathcal{U}(K_\lambda)$

The following result shows a very close connection between inner parallel bodies and form bodies for which we refer to Schneider [11].

Theorem 2.2 (Schneider [11, Lemma 3.1.10])**.** *Let $K \in \mathcal{K}^n$. Then K_λ is a dilation of K, for all $-\mathrm{r}(K) < \lambda \leq 0$ if and only if $K = \mathrm{r}(K)K^{\mathbb{1}}$.*

There exist also strong relations between inner parallel bodies, form bodies and extreme vectors through the so called Riemann-Minkowski integral (see [4] and [9, Lemma 3.2]). For a convex body K with inradius $\mathrm{r}(K)$, the Riemann-Minkowski integral of $(K_\lambda)^{\mathbb{1}}$ in $-\mathrm{r}(K) \leq \lambda \leq 0$, $\int_{-\mathrm{r}}^{0} K_\lambda^{\mathbb{1}} d\lambda$ is the convex body whose support function is given by

$$h\left(\int_{-\mathrm{r}}^{0} K_\lambda^{\mathbb{1}} d\lambda, u\right) = \int_{-\mathrm{r}(K)}^{0} h(K_\lambda^{\mathbb{1}}, u)\, d\lambda, \quad \text{for all } u \in \mathrm{S}^{n-1}.$$

Theorem 2.3 (Sangwine-Yager [9, Lemma 3.2.])**.** *Let $K \in \mathcal{K}^n$. Then*

$$K \supseteq K_{-\mathrm{r}(K)} + \int_{-\mathrm{r}}^{0} K_\lambda^{\mathbb{1}} d\lambda.$$

Equality holds if

$$\mathrm{cl}\left(\mathcal{U}(K_\lambda)\right) = \mathrm{cl}(\mathcal{U}(K_\lambda + K_\lambda^{\mathbb{1}})) \tag{2.1}$$

for $-\mathrm{r}(K) < \lambda \leq 0$.

Equation (2.1) plays an important role in Theorem 5.2 and consequently in Theorems 1.1 and 1.2. Furthermore, in dimension two, as we shall see in Section 4, (2.1) holds for any convex body and the above inclusion is always an equality. The geometry behind (2.1) is not completely understood. Sufficient conditions for a convex body to satisfy this condition are not known. However, for a polytope P, as Theorem 5.2 and Corollary 5.3 show, this condition is equivalent to a precise decomposition of P using the form bodies of its inner parallel bodies.
We would like to mention also the following theorem which also deals with decomposition of convex bodies through their inner parallel bodies.

Theorem 2.4 (Hernández Cifre, Saorín Gómez, [8, Theorem 2.2])**.** *Let $K \in \mathcal{K}^n$. Then $K = K_\lambda + |\lambda|\, K^{\mathbb{1}}$ for every $-\mathrm{r}(K) \leq \lambda \leq 0$ if and only if*

$$h(K,u) = h(K_{-\mathrm{r}(K)}, u) + \mathrm{r}(K), \text{ for all } u \in \mathcal{U}(K) \tag{2.2}$$

and $\mathrm{cl}\,\mathcal{U}(K) = \mathcal{U}(K_\lambda + K^{\mathbb{1}})$ *for all* $-\mathrm{r}(K) \leq \lambda \leq 0$.

A convex body K satisfying (2.2) is called a tangential body of $K_{-\,\mathrm{r}(K)} + \mathrm{r}(K)\,\mathrm{B}_n$. For precise definitions and further information about tangential bodies we refer to Schneider [11, Section 2.2]. We would like to point out, that for the above result it is necessary to assume that the body K have both, K_λ and $K^{\mathbb{1}}$ as summands (cf. Proposition 3.6 and Corollary 3.7).

3. Auxillary results

The following two corollaries are immediate consequences of Theorem 2.2. A summand of a convex body K is said to be *trivial* if it is a dilation of K.

Corollary 3.1. *If $P = \mathrm{r}(P)P^{\mathbb{1}}$, then all inner parallel bodies of P are (trivial) summands of P.*

A convex body $K \in \mathcal{K}^n$ is indecomposable if all its summands are trivial, i.e., dilations of itself. For example, simplicial polytopes or pyramids (see e.g. [12, Section 15.1]) are indecomposable while simple polytopes (except for the simplex) are not.

Corollary 3.2. *Let P be indecomposable, $-\mathrm{r}(P) \leq \tau \leq 0$. Then P_τ is a summand of P if and only if $P = \mathrm{r}(P)P^{\mathbb{1}}$.*

The following criterion gives necessary and sufficient conditions in order a polytope to be a summand of another one.

Theorem 3.3 (Shepard's decomposition criterion [12])**.** *Q is summand of P if and only if the following two conditions hold:*

(i) $\dim(F(P,u)) \geq \dim(F(Q,u))$ *for every* $u \in S^{n-1}$.

(ii) *For every edge $F(P,u)$ of P, it is*

$$\mathrm{vol}_1(F(P,u)) \geq \mathrm{vol}_1(F(Q,u)).$$

Let P be a polytope and let F be a face of P. The set

$$N(F) = \mathrm{cl}\left(\mathrm{cone}\{u \in \mathrm{S}^{n-1} : F(P,u) = F\}\right)$$

that consists of all vectors $u \in \mathrm{S}^{n-1}$ that are normal to F is called the normal cone of F. The poset of all normal cones of P, ordered by inclusion, is called the normal fan of P, denoted by $\mathcal{N}(P)$. That is for a non-empty polytope in $\mathbb{R}^n$, $\mathcal{N}(P)$ consists of the normal cones of all faces of P. The union of all such cones is $\mathbb{R}^n$, which means that $\mathcal{N}(P)$ is a complete fan and furthermore, the relative interiors of the cones in the normal fan form a partition of $\mathbb{R}^n$ (see [13, Section 7.1] for a more detailed introduction).
Lemma 2.1 (i) provides a relation between the 0-dimensional elements of $\mathcal{N}(P)$ and $\mathcal{N}(P_\tau)$. For the other dimensional cones in $\mathcal{N}(P)$ and $\mathcal{N}(P_\tau)$ no analogous relation holds in general. However, if we ask P_τ to be a summand of P, using Shephard's decomposition criterion (Theorem 3.3) the following well known result can be proved.

Proposition 3.4. *If P_τ is a summand of P, then the normal fan of P is a refinement of the normal fan of P_τ. The converse is not true (see* Proposition 5.1 (iii)).

If $P \in \mathcal{K}^n$ is a polytope, $\mathcal{U}(P)$ is the set of outer normal vectors to the facets of P, which coincide with the 1-dimensional cones in the normal fan. This ensures that there exists $\epsilon > 0$ so that, for $-\epsilon < \lambda \leq 0$, $\mathcal{U}(P_\lambda) = \mathcal{U}(P)$, i.e., there is a range in $(-\mathrm{r}(P), 0]$ in which the polytopes P_λ have exactly the same number of facets as P does. Notice that this is no longer true for a general convex body; see e.g. [9, Figure 2.3].
Using this, we define the following parameters, $\tau_j(P)$, for $j \in \mathbb{N}$, associated to P.

Definition 3.5. Let $\tau_0(P) = 0$, $\tau_1(P) = \inf\{\mu \in (-\mathrm{r}(P), 0] : \mathcal{U}(P_\mu) = \mathcal{U}(P)\}$. Inductively, let $\tau_i(P) = \tau_1(P_{\tau_{i-1}(P)}) = \inf\{\mu \in (-\mathrm{r}(P), 0] : \mathcal{U}(P_\mu) = \mathcal{U}(P_{\tau_{i-1}(P)})\}$.

Taking into account that $P_{-\mathrm{r}(P)}$ has dimension strictly less than n, Lemma 2.1 (i) and the previous comments, it is clear that there exist only finitely many (different) $\tau_i(P)$. Notice, that if the polytope P has no interior points, i.e., if its inradius is 0, then $\tau_1(P) = 0$. Observe also that $\tau_1(P)$ is not a minimum, i.e., $\mathcal{U}(P_{\tau_1(P)}) \neq \mathcal{U}(P)$. Indeed $\tau_1(P)$ can be described geometrically as the largest value on the interval $(-\mathrm{r}(P), 0]$ for which $P_{\tau_1(P)}$ has strictly less facets than P. Hence, $\tau_1(P) = -\mathrm{r}(P)$ if and only if

$$h(P_{-\mathrm{r}(P)}, u) = h(P, u) - \mathrm{r}(P), \quad \text{for all } u \in \mathcal{U}(P).$$

Next we prove that for $\lambda \in [-\tau_1(P), 0]$ there are necessary and sufficient conditions in order P_λ to be a summand of P and these rely on the form body of P. Outside this interval, it will be necessary that P_λ, for all λ in at least some open interval of $(-\mathrm{r}(P), 0]$ are summands of P, in order to prove our decomposability conditions.

Proposition 3.6. *Let $\tau_1(P) \leq \tau \leq 0$ and let P_τ be a summand of P. Then $P = P_\tau + |\tau| P^{\mathbb{1}}$.*

Proof. Let $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq b_i,\ 1 \leq i \leq m\}$ for $u_i \in \mathrm{S}^{n-1}$, $b_i \in \mathbb{R}$ and $1 \leq i \leq m$. Then

$$P_\tau = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq b_i - |\tau|,\ 1 \leq i \leq m\}.$$

For $\tau_1(P) < \tau \leq 0$, we have that $u_i \in \mathcal{U}(P_\tau)$ for $1 \leq i \leq m$. Thus, from Lemma 2.1 (vi) follows that $h(P, u) = h(P_\tau, u) + |\tau|$. The continuity of the support function ensures that this relation holds for $\tau = \tau_1(P)$ too.
Let Q be so that $Q + P_\tau = P$. Then Q is the Minkowski difference of P and P_τ and therefore we can write

$$\begin{aligned} Q &= \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq h(P, u_i) - h(P_\tau, u_i),\ 1 \leq i \leq m\} \\ &= \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq |\tau|,\ 1 \leq i \leq m\} \\ &= |\tau|\, P^{\mathbb{1}} \end{aligned}$$

for any $\tau_1(P) \leq \tau \leq 0$ where we have implicitly used Lemma 2.1 (ii), i.e., $\mathcal{U}(Q) = \mathcal{U}(P \sim P_\lambda) \subseteq \mathcal{U}(P)$. ∎

Proposition 3.6 provides a slight improvement of Theorem 2.4, namely, it is not necessary to ask for the precise decomposition of P, but just for P_τ to be a summand of P for all $-\mathrm{r}(P) \leq \tau \leq 0$.

Corollary 3.7. *Let $P \in \mathcal{K}^n$ be a polytope with $\mathrm{r}(P) = \tau_1(P)$. The following conditions are equivalent:*

(i) *P_τ is a summand of P for all $-\mathrm{r} \leq \tau \leq 0$.*
(ii) *$\mathcal{U}(P) = \mathcal{U}(P_\tau + P^{\mathbb{1}})$ for all $-\mathrm{r}(P) \leq \tau \leq 0$.*

Proof. Since $\tau_1(P) = \mathrm{r}(P)$, from Proposition 3.6 we obtain that $P = P_\tau + |\tau|\, P^{\mathbb{1}}$ for all $-\mathrm{r}(P) \leq \tau \leq 0$, it is clear that $\mathcal{U}(P) = \mathcal{U}(P_\tau + P^{\mathbb{1}})$ for all $-\mathrm{r}(P) \leq \tau \leq 0$.
For the converse, since $\tau_1(P) = \mathrm{r}(P)$, $u \in \mathcal{U}(P) = \mathcal{U}(P_\tau)$ for all $-\mathrm{r}(P) < \tau \leq 0$, it follows that $h(P,u) = h(P_\tau, u) + |\tau|\, h(P^{\mathbb{1}}, u)$ for $u \in \mathcal{U}(P)$. Condition $\mathcal{U}(P) = \mathcal{U}(P_\tau + P^{\mathbb{1}})$ yields that in fact $P = P_\tau + |\tau|\, P^{\mathbb{1}}$. □

We notice, that the normal cones of P and $P^{\mathbb{1}}$ are, in general, no refinements one of the other (see [9, Figure 2.2]).

Corollary 3.8. *If P_τ is a summand of P for some $\tau_1(P) \leq \tau \leq 0$ then the normal fan of P is a refinement of the normal fan of $P^{\mathbb{1}}$. The converse is not true (see* Figure 4).

If all the inner parallel bodies of P in the range $\tau_1(P) \leq \tau \leq 0$ are summands of P, we get from Proposition 3.6 that $P = P_{\tau_1(P)} + \tau_1(P) P^{\mathbb{1}}$. This allows to provide more information on the normal fans of P and P_τ for $\tau_1(P) < \tau \leq 0$, improving Proposition 3.6. For, we need the following lemma.

Lemma 3.9. *Let $P = P_\tau + |\tau| P^{\mathbb{1}}$ for some $\tau_1(P) \leq \tau \leq 0$. Then $P_\mu = \left(1 - \frac{|\mu|}{|\tau|}\right) P + \frac{|\mu|}{|\tau|} P_\tau$ for all $\tau \leq \mu \leq 0$.*

Proof. Let $P = P_\tau + |\tau| P^{\mathbb{1}}$ for some $\tau_1(P) \leq \tau \leq 0$. With

$$\begin{aligned} h(P_\mu, u) + |\mu| h(P^{\mathbb{1}}, u) &= h(P_\mu + |\mu| P^{\mathbb{1}}, u) \leq h(P, u) \\ &= h(P_\tau + |\tau| P^{\mathbb{1}}, u) = h(P_\tau, u) + |\tau| h(P^{\mathbb{1}}, u) \end{aligned}$$

we follow $P_\mu \subset P_\tau + (|\tau| - \mu|) P^{\mathbb{1}} \subset P_\mu$, which implies that $P_\mu = P_\tau + (|\tau| + \mu) P^{\mathbb{1}}$ and $P = P_\mu + |\mu|\, P^{\mathbb{1}}$ for $\tau \leq \mu \leq 0$. Thus,

$$\begin{aligned} \frac{|\mu|}{|\tau|} P_\tau + \left(1 - \frac{|\mu|}{|\tau|}\right) P &= \frac{|\mu|}{|\tau|} P_\tau + \left(1 - \frac{|\mu|}{|\tau|}\right) \left(P_\mu + |\mu|\, P^{\mathbb{1}}\right) \\ &= \frac{|\mu|}{|\tau|} P_\tau + \left(\frac{|\tau| - |\mu|}{|\tau|}\right) P_\mu + \frac{|\mu|}{|\tau|} \big(|\tau| - |\mu|\big) P^{\mathbb{1}} \\ &= \frac{|\mu|}{|\tau|} \Big[P_\tau + \big(|\tau| - |\mu|\big) P^{\mathbb{1}}\Big] + \frac{|\tau| - |\mu|}{|\tau|} P_\mu \\ &= \frac{|\mu|}{|\tau|} P_\mu + \frac{|\tau| - |\mu|}{|\tau|} P_\mu = P_\mu. \end{aligned}$$

□

Remark 3.10. The proof of Lemma 3.9 shows in particular, that, if P_τ is a summand of P for some $\tau_1(P) \leq \tau \leq 0$, then P_μ is a summand of P as well for all $\tau \leq \mu \leq 0$.

Together with Proposition 3.4 we can say more about the normal fan of a polytope, some of whose inner parallel bodies are summands of it:

Corollary 3.11. *If P_τ is a summand of P for all $\tau_1(P) \leq \tau \leq 0$, then the normal fans of P and P_τ coincide for all $\tau_1(P) < \tau \leq 0$.*

4. Decompositions in dimension 2

In this section we prove that every convex polygon can be written as the sum of its kernel and the Riemann-Minkowski integral of the form bodies of its inner parallel bodies. In order to do it, we establish first the following lemma. Although its proof can be deduced from Lemma 3.2.3 in [11], we include it here for completeness.

Lemma 4.1. *Let P be a convex polygon, $-\mathrm{r}(P) \leq \tau \leq 0$. Then P_τ is a summand of P.*

Proof. We use Shephard's criterion. The first condition is obvious. For the second condition, let $u \in \mathrm{S}^{n-1}$ such that $F(P,u)$ is an edge of P. Then $u \in \mathcal{U}(P) \supset \mathcal{U}(P_\tau)$. If $u \notin \mathcal{U}(P_\tau)$, $F(P_\tau,u)$ is a vertex an the condition is fulfilled. Let otherwise $u \in \mathcal{U}(P_\tau)$. Then with $h(P_\tau,u) = h(P,u) + \tau$ (see Lemma 2.1) and $F(P_\tau,u) + |\tau|\,\mathrm{B}_n \subset P$ it follows, that

$$\begin{aligned} F(P,u) &\supset (F(P_\tau,u) + |\tau|\,\mathrm{B}_n) \cap H(P,u) \\ &= (F(P_\tau,u) + |\tau|\,\mathrm{B}_n) \cap (H(P_\tau,u) + |\tau| u) \\ &= F(P_\tau,u) + |\tau| u, \end{aligned}$$

which implies the second condition. □

Notice that, the above lemma implies, in particular, that if P_τ is a summand of P, then P_τ is also a summand of P_μ for all $\tau \leq \mu \leq 0$, because from the definition of inner parallel bodies it is clear that $(P_\tau) = (P_\mu)_{\tau-\mu}$. In Proposition 5.1 and Corollary 5.3 we provide examples of polytopes P all whose inner parallel bodies are summands of them, i.e., P_μ is a summand of P, for $-\mathrm{r}(P) \leq \mu \leq P$, but P_μ is not a summand of P_τ for some $\mu < \tau < 0$. In the next result we prove an explicit decomposition of any convex polygon P through some of its inner parallel bodies P_τ, $-\mathrm{r}(P) \leq \tau \leq 0$. Although this result is a consequence of Theorem 2.3 (see also the comments below Theorem 2.3 about the planar case), we provide the proof for the case of polygons since the same argument works for the general case in the proof of Theorem 5.2.

Theorem 4.2. *Let P be a convex polygon, let $i \in \mathbb{N}$ with $\tau_{i+1}(P) \leq \tau \leq \tau_i(P)$. Then*

$$P = P_\tau + |\tau - \tau_i(P)|(P_{\tau_i(P)})^{\mathbb{1}} + \sum_{j=1}^{i} |\tau_j(P) - \tau_{j-1}(P)|(P_{\tau_{j-1}(P)})^{\mathbb{1}}.$$

Proof. Let $\tau_{i+1}(P) \leq \tau \leq \tau_i(P)$. Then by Lemma 4.1, we get that P_τ is a summand of $P_{\tau_i(P)}$. By Proposition 3.6, we have

$$P_{\tau_i(P)} = P_\tau + |\tau_i(P) - \tau| P^{\mathbb{1}}_{\tau_i(P)}.$$

Again by Lemma 4.1 and Proposition 3.6, $P_{\tau_i(P)}$ is a summand of $P_{\tau_{i-1}(P)}$ and

$$\begin{aligned} P_{\tau_{i-1}(P)} &= P_{\tau_i(P)} + |\tau_{i-1}(P) - \tau_i(P)| P^{\mathbb{1}}_{\tau_{i-1}(P)} \\ &= P_\tau + |\tau_i(P) - \tau| P^{\mathbb{1}}_{\tau_i(P)} + |\tau_{i-1}(P) - \tau_i(P)| P^{\mathbb{1}}_{\tau_{i-1}(P)}. \end{aligned}$$

Repeating this argument $i-1$ times, yields the theorem. $\square$

Since $P^{\mathbb{1}}_\tau$ is constant for $\tau \in (\tau_{i+1}(P), \tau_i(P)]$, this theorem is in fact equivalent to the following result in which the numbers $\tau_i(P)$ are replaced by a Riemann-Minkowski integral.

Corollary 4.3. *Let P be a convex polygon, $-\mathrm{r}(P) \leq \tau \leq 0$. Then*

$$P = P_\tau + \int_\tau^0 (P_\mu)^{\mathbb{1}} d\mu.$$

5. Decompositions in dimension n

First we prove that, unlike in dimension 2, for $n \geq 3$ inner parallel bodies of a convex body may not all be summands of it. This fact will amount to a drastically different behavior of the summands of a polytope.

Proposition 5.1. *Let $n \geq 3$.*

(i) *There are n-dimensional polytopes, all of whose inner parallel bodies are summands of them.*
(ii) *There are n-dimensional polytopes, some of whose inner parallel bodies are summands of them.*
(iii) *There are n-dimensional polytopes, non of whose inner parallel bodies are summands of them.*

Proof. For $c \in [\frac{20}{3}, 12]$ let

$$P(c) = \left\{ x \in \mathbb{R}^3 : \begin{array}{rcl} \pm 12x_1 + 35x_3 & \leq & 432, \\ \pm 12x_2 + 5x_3 & \leq & 60, \\ x_3 & \geq & 0, \\ x_3 & \leq & c \end{array} \right\}$$

(see Figure 1. To ensure perspicuity, the x_1-axis is dilated by $\frac{1}{2}$ in all pictures in this proof). The inradius is $\mathrm{r}(P(c)) = \frac{10}{3}$. The inner parallel bodies for

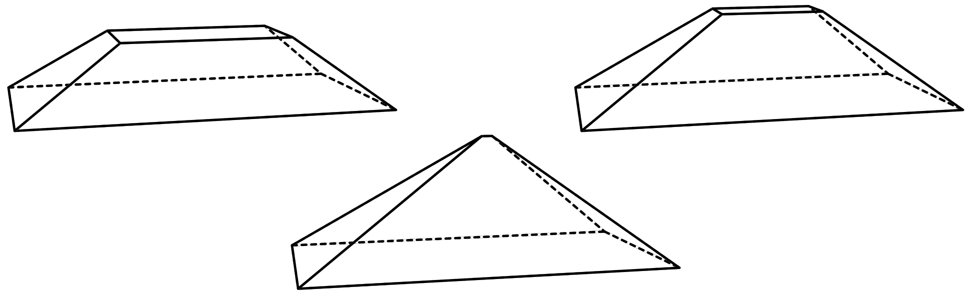

FIGURE 1. $P(\frac{20}{3})$, $P(9)$ and $P(12)$; (to ensure perspicuity, the x_1-axis is dilated by $\frac{1}{2}$ in all pictures in this proof)

$-\mathrm{r}(P) \le \tau \le 0$ are given by

$$P(c)_\tau = \left\{ x \in \mathbb{R}^3 : \begin{array}{rcl} \pm 12x_1 + 35x_3 & \le & 432 + 37\tau, \\ \pm 12x_2 + 5x_3 & \le & 60 + 13\tau, \\ x_3 & \ge & 0 - \tau, \\ x_3 & \le & c + \tau \end{array} \right\}$$

and

$$P(c)_{-\frac{10}{3}} = \operatorname{conv}\left\{ \left(\pm 16, 0, \frac{10}{3}\right)^\top \right\}.$$

Furthermore, $\tau_1(P(c)) = -\frac{60-5c}{8}$.
Altogether, we have

$$P(c)_\tau = \operatorname{conv}\left\{ \begin{pmatrix} \pm(36-6\tau) \\ \pm(5-\frac{3}{2}\tau) \\ \tau \end{pmatrix}, \begin{pmatrix} \pm(36-\frac{35}{12}c-\frac{1}{6}\tau) \\ \pm(5-\frac{5}{12}c-\frac{2}{3}\tau) \\ c-\tau \end{pmatrix} \right\}$$

for $\tau \in (\tau_1(P(c)), 0]$ and

$$P(c)_\tau = \operatorname{conv}\left\{ \begin{pmatrix} \pm(36-6\tau) \\ \pm(5-\frac{3}{2}\tau) \\ \tau \end{pmatrix}, \begin{pmatrix} \pm(1+\frac{9}{2}\tau) \\ 0 \\ 12-\frac{13}{5}\tau \end{pmatrix} \right\}$$

for $\tau \in (\frac{10}{3}, \tau_1(P(c))]$.
It is clear that all inner parallel bodies satisfy the first condition in Shephard's theorem. To check the second condition, the length of the upper edges with direction $(1,0,0)^\top$ is of importance.

(i) Let $c \in [-\frac{20}{3}, -\frac{48}{7}]$. Then it is easy to check, that $P(c)_\tau$ satisfies the second condition in Shephard's theorem for all $\tau \in [-\frac{10}{3}, 0]$ (see Figure 2).

(ii) Let $c \in (-\frac{48}{7}, 12)$. Then it is easy to check, that $P(c)_\tau$ satisfies the second condition in Shephard's theorem for all $\tau \in [-\frac{70}{9}+\frac{35}{54}c, 0]$ and $P(c)_\tau$ does not satisfy the second condition in Shephard's theorem for all $\tau \in [-\frac{10}{3}, -\frac{70}{9}+\frac{35}{54}c)$ (see Figure 3).

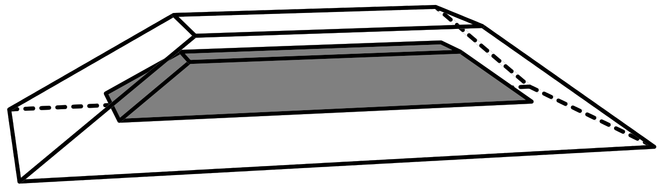

FIGURE 2. $P(\frac{20}{3})$ and $P(\frac{20}{3})_{\frac{40}{21}}$

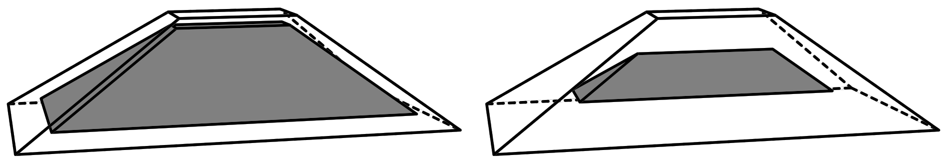

FIGURE 3. $P(9)$, $P(9)_{\frac{20}{21}}$ and $P(9)_{\frac{50}{21}}$

(iii) Let $c = 12$. Then for all $\tau \in [-\frac{10}{3}, 0)$, $P(c)_\tau$ does not satisfy the second condition in Shephard's theorem (see Figure 4).

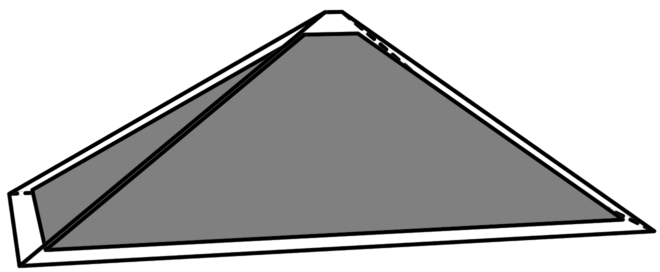

FIGURE 4. $P(12)$ and $P(12)_{\frac{10}{21}}$

□

Next we will prove Theorems 1.1 and 1.2 which are consequences of the following more general result.

Theorem 5.2. *Let* $-\mathrm{r}(P) \leq \mu_1 < \mu_2 \leq 0$. *The following statements are equivalent:*

(i) P_τ *is summand of* $P_{\tilde{\tau}}$, *for all* $\mu_1 \leq \tau \leq \tilde{\tau} \leq \mu_2$.
(ii) $h(P_{\mu_2}, u) = h(P_\tau, u) + \int_\tau^{\mu_2} h((P_\mu)^{\mathbb{1}}, u) d\mu$, *for all* $u \in \mathrm{S}^{n-1}$ *and for all* $\mu_1 \leq \tau \leq \mu_2$.
(iii) $\mathcal{U}(P_\tau + (P_\tau)^{\mathbb{1}}) = \mathcal{U}(P_\tau)$, *for all* $\mu_1 \leq \tau \leq \mu_2$.
(iv) $\frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau} = h((P_\tau)^{\mathbb{1}}, u)$, *for all* $\mu_1 \leq \tau \leq \mu_2$ *for which the derivative exists for all* $u \in \mathrm{S}^{n-1}$.
(v) $u \mapsto \frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau}$ *is a support function, for all* $\mu_1 \leq \tau \leq \mu_2$ *for which the derivative exists for all* $u \in \mathrm{S}^{n-1}$.

Notice that Theorem 1.1 is exactly the step (i) $\Leftrightarrow$ (ii) and Theorem 1.2 is (i) $\Leftrightarrow$ (iii), in both cases for $\mu_2 = 0$.

Proof. We show (i) $\Leftrightarrow$ (ii), (iv) $\Leftrightarrow$ (v), and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

- (i) $\Leftrightarrow$ (ii): The proof is analogous to the corresponding proofs in dimension 2 (See Theorem 4.2 and Corollary 4.3).
- (i) $\Rightarrow$ (iii): Let $\mu_1 \leq \tau \leq \mu_2$, and let $i \in \mathbb{N}$, such that $\tau_{i+1}(P) \leq \tau \leq \tau_i(P)$. Then by Proposition 3.6 and (i) we have $P_{\tau_i(P)} = P_\tau + |\tau_i(P) - \tau|(P_{\tau_i(P)})^{\mathbb{1}} = P_\tau + |\tau_i(P) - \tau|(P_\tau)^{\mathbb{1}}$. Thus $\mathcal{U}(P_\tau) = \mathcal{U}(P_{\tau_i(P)}) = \mathcal{U}(P_\tau + (P_\tau)^{\mathbb{1}})$.
- (iii) $\Rightarrow$ (iv): This follows directly from Theorem 2.3.
- (iv) $\Rightarrow$ (ii): This is immediate by integrating from τ to μ_2.
- (iv) $\Leftrightarrow$ (v): Assume (v) and let $\mu_1 \leq \tau \leq \mu_2$, such that the derivative exists. Then $\frac{d}{d\mu}h(P_\mu, \cdot)|_{\mu=\tau}$ is a support function. Hence, let R_τ be the convex body with support function $\frac{d}{d\mu}h(P_\mu, \cdot)|_{\mu=\tau}$. Since P_μ is a polytope, $h(P_\mu, u)$ is linear in u in all full-dimensional cones in the normal fan $\mathcal{N}(P_\mu)$ and thus, the same is true for $u \mapsto \frac{d}{d\mu}h(P_\mu, \cdot)|_{\mu=\tau}$. Since support functions of polytopes are characterized by being piecewise linear support functions of convex bodies (see [6, Exercise 3.1.19.]), R_τ is a polytope and its normal fan is only coarser than that of P_τ, i.e., $\mathcal{U}(R_\tau) \subset \mathcal{U}(P_\tau) = \mathcal{U}(P_\tau^{\mathbb{1}})$. To show (iv) it remains to show $h(R_\tau, u) = h(P_\tau^{\mathbb{1}}, u)$ for all $u \in \mathcal{U}(P_\tau^{\mathbb{1}})$, i.e., $\frac{d}{d\mu}h(P_\mu, \cdot)|_{\mu=\tau} = 1$ for all $u \in \mathcal{U}(P_\tau^{\mathbb{1}})$. This is true since for $u \in \mathcal{U}(P_\tau^{\mathbb{1}})$ we have $h(P_\mu, u) = h(P, u) + \mu$ and thus $\frac{d}{d\mu}h(P_\mu, u) = 1$. The converse direction is trivial.

□

We would like to remark that this result contains the converse of Theorem 2.3 for the case of polytopes, namely, it provides necessary conditions for a polytope to have equality in Theorem 2.3.
The assertion in the following corollary is a direct consequence of Theorem 5.2 as well as Theorem 2.3.

Corollary 5.3. *Let $\mathcal{U}(P_\tau + (P_\tau)^{\mathbb{1}}) = \mathcal{U}(P_\tau)$ for all $-\operatorname{r}(P) \leq \tau \leq 0$. Then P_τ is a summand of P for all $-\operatorname{r}(P) \leq \tau \leq 0$. The converse is not true!*

Proof. By Theorem 5.2 (or Theorem 2.3), $\mathcal{U}(P_\tau + (P_\tau)^{\mathbb{1}}) = \mathcal{U}(P_\tau)$ for all $-\operatorname{r}(P) \leq \tau \leq 0$ implies, that P_τ is summand of $P_{\tilde{\tau}}$, for all $-\operatorname{r}(P) \leq \tau \leq \tilde{\tau} \leq 0$ which proves the assertion. For the converse, let $P(c)$ be as in proof of Proposition 5.1 and let $c \in (-\frac{20}{3}, -\frac{48}{7}]$. Then $P(c)_\tau$ is a summand of $P(c)$ for all $\tau \in [-\frac{10}{3}, 0]$. However, $P(c)_\tau$ is not a summand of $P(c)_{\tau(P(c))}$, whenever $\tau \leq \tau(P(c))$ and thus by Theorem 5.2, $\mathcal{U}(P_\tau + (P_\tau)^{\mathbb{1}}) = \mathcal{U}(P_\tau)$ cannot be fulfilled for all $-\operatorname{r}(P) \leq \tau \leq 0$. □

6. Moving facets outwards

In this section, we have a brief look at a similar question, if we move facets of the polytope outwards.

For a polytope $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq b_i, 1 \leq i \leq m\}$ with unit normals to the facets $u_i \in \mathrm{S}^{n-1}$ and $\tau > 0$ we denote by $P_\tau = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq b_i + \tau, 1 \leq i \leq m\}$. We remark that this notion differs from the so-called *outer parallel body* of a convex body. Nonetheless with this notation, we have that $(P_\tau)_\mu = P_{\tau+\mu}$ for all $\tau > -\mathrm{r}(P)$ and $\mu > -(\mathrm{r}(P)+\tau)$. This also implies, that $\tau_i(P_\tau) = \tau_i(P) + \tau$ for all $\tau > \tau_1(P)$ and $\mathrm{r}(P_\tau) = \mathrm{r}(P) + \tau$ for all $\tau > -\mathrm{r}(P)$.

In the case of positive τ, the combinatorial properties of P_τ are easier than for negative τ:

Lemma 6.1. *Let $\tau > 0$. Then $\mathcal{U}(P) = \mathcal{U}(P_\tau)$.*

Proof. Assume w.l.o.g. $F(P_\tau, u_1)$ is not a facet, i.e., $\langle x, u_i \rangle \leq b_i + \tau$ is redundant. Hence there are $\alpha_i > 0$, $2 \leq i \leq m$ with $\sum_{i=2}^m \alpha_i u_i = u_1$ and $\sum_{i=2}^m \alpha_i (b_i + \tau) \leq b_1 + \tau$. Thus $\sum_{i=2}^m \alpha_i b_i + (m-1)\tau \leq b_1 + \tau$ which is $\sum_{i=2}^m \alpha_i b_i \leq b_1 - (m-2)\tau \leq b_1$. This is a contradiction, since $F(P, u_1)$ is a facet of P. □

We want to answer the question, if and when P is a summand of P_τ, $\tau > 0$. Since, as Lemma 6.1 shows, P_τ has the same facet normals as P for all $\tau > 0$, the situation is altogether similar and based on the situation in the interval $[\tau_1(P), 0]$.

Theorem 6.2. *The following statements are equivalent:*

(i) *P is a summand of P_τ for some $\tau > 0$.*
(ii) *P is a summand of P_τ for all $\tau \geq 0$.*
(iii) *P is a nested summand of P_τ for all $\tau \geq 0$, i.e., P_μ is a summand of P_τ for all $\tau > \mu \geq 0$.*
(iv) *$\mathcal{U}(P + P^{\mathbb{1}}) = \mathcal{U}(P)$.*
(v) *$P_\tau = P_\mu + (\tau - \mu) P^{\mathbb{1}}$ for all $\tau > \mu \geq 0$.*

Proof. We show (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) and the step from (v) $\Rightarrow$ (i) is obvious.

- (i) $\Rightarrow$ (iv): Let P be a summand of P_τ. By Lemma 6.1 $\tau_1(P_\tau) < -\tau$ and thus by Proposition 3.6 we have $P_\tau = (P_\tau)_{-\tau} + \tau P_\tau^{\mathbb{1}} = P + \tau P^{\mathbb{1}}$ which implies (iv) since $\mathcal{U}(P) = \mathcal{U}(P_\tau) = \mathcal{U}(P + P^{\mathbb{1}})$.
- (iv) $\Rightarrow$ (ii): This follows from Theorem 2.3.
- (ii) $\Rightarrow$ (iii): Let P be a summand of both, P_τ and P_μ with $\tau > \mu > 0$. Then by Lemma 6.1 and Proposition 3.6, we have $h(P_\tau, u) = h(P, u) + \tau h(P^{\mathbb{1}}, u)$ and $h(P_\mu, u) = h(P, u) + \mu h(P^{\mathbb{1}}, u)$ for all $u \in \mathrm{S}^{n-1}$. Subtraction of both equations yields $h(P_\tau, u) = h(P_\mu, u) + (\tau - \mu) h(P^{\mathbb{1}}, u)$ for all $u \in \mathrm{S}^{n-1}$ which implies (iii).

- (iii) $\Rightarrow$ (v): Let $\tau > \mu \geq 0$. Then $P_\mu = (P_\tau)_{\mu-\tau}$ is a summand of P_τ. Since $0 > \mu - \tau \geq -\tau > \tau_1(P_\tau)$ we get by Proposition 3.6 $P_\tau = (P_\tau)_{\mu-\tau} + |\mu - \tau| P_\tau^{\mathbb{1}} = P_\mu + (\tau - \mu) P^{\mathbb{1}}$.

$\square$

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